Lab 10. Maximum likelihood method.

9.1 $X: \begin{pmatrix} 3 & 7 \\ \theta & 1-\theta \end{pmatrix}$ $\quad P(X=3)=\theta$

$P(X=7)=1-\theta$

$X \sim Bern(\theta)$

(a) Method of moments : $\boxed{\mu_1 = m_1} \Leftrightarrow 7-4\theta = 4.5 \Leftrightarrow 4\theta = 2.5 \Leftrightarrow \hat{\theta} = 0.625$

$\mu_1 = E[X]$

$\mu_1 = \frac{1}{n}\sum_{i=1}^{n} X_i \Leftrightarrow 7-4\theta = \frac{1}{n}\sum X_i \Leftrightarrow \hat{\theta} = \frac{7}{4} - \frac{1}{4n}\sum X_i$

m_1 = sample mean

$V[\hat{\theta}] = \frac{1}{(4n)^2}\sum_{i=1}^{n} V[X_i]$

3, 3, 3, 3, 3, 7, 7, 7 - sample

$\mu_1 = E[X] = 3\theta + 7\cdot(1-\theta) = 3\theta + 7 - 7\theta = 7-4\theta$

$m_1 = \frac{3\cdot5+7\cdot3}{8} = \frac{15+21}{8} = \frac{36}{8} = 4.5$

(b) Method of maximum likelihood:

$P(X) = \prod_{i=1}^{8} P(X_i) = \theta^5 \cdot (1-\theta)^3 \Rightarrow \ln P(X) = 5\ln\theta + 3\ln(1-\theta)$ - maximum

$P(X_i) = P(X_i = x_i), \ i = \overline{1,8}$

$P(X_1) = P(X_1 = 3) = \theta$

$P(X) = P((X_1, X_2, \dots, X_8) = (3,3,3,3,3,7,7,7))$

$\frac{d \ln P(X)}{d\theta} = \frac{5}{\theta} - \frac{3}{1-\theta} = 0 \Leftrightarrow \frac{5}{\theta} = \frac{3}{1-\theta} \Leftrightarrow 5(1-\theta) = 3\theta \Leftrightarrow 5 - 5\theta = 3\theta \Leftrightarrow$

$8\theta = 5 \Leftrightarrow \underline{\hat{\theta} = 0.4}$

c) $SD[\hat{\theta}] = ?$

$V[\hat{\theta}] = ?$

8.2. X - nr. of times a computer code is executed until it runs without error

$X \sim Geom(p)$

$$X : \begin{pmatrix} 1 & 2 & 3 & \dots & k & \dots \\ & & \dots & & (1-p)^{k-1}\cdot p & \dots \end{pmatrix} \qquad P(X=k) = (1-p)^{k-1}\cdot p$$

3, 7, 5, 3, 2 - sample

(a) Method of moments : $\mu_1 = m_1 \iff \frac{1}{p} = 4 \iff \hat{p} = \frac{1}{4}$; $\underline{\hat{p} = 0{,}25}$

$\mu_1 = E[X] = \frac{1}{p}$

m_1 - sample mean

$$m_1 = 4 = \frac{3+7+5+3+2}{5}$$

(b) Method of maximum likelihood

$$P(X) = P((X_1, X_2, X_3, X_4, X_5) = (3,7,5,3,2)) = P(X_1=3)\cdot P(X_2=7)\cdot P(X_3=5)\cdot$$
$$\cdot P(X_4=3)\cdot P(X_5=2)$$

$$P(X) = \underbrace{(1-p)^2\cdot p}\cdot \underbrace{(1-p)^6\cdot p}\cdot \underbrace{(1-p)^4\cdot p}\cdot \underbrace{(1-p)^2\cdot p}\cdot \underbrace{(1-p)\cdot p} = p^5(1-p)^{15}$$

$X_i \sim Geom(p)$

$$L(p) = \ln P(X) = \ln\left[p^5\cdot(1-p)^{15}\right] = 5\ln p + 15\ln(1-p) \quad - \text{maximum}$$

$$\frac{dL(p)}{dp} = \frac{5}{p} - \frac{15}{1-p} = 0 \iff \frac{\cancel{5}}{p} = \frac{\cancel{15}^{\,3}}{(1-p)} \iff 1-p = 3p \iff 1 = 4p \iff \hat{p} = \frac{1}{4}$$

3.1. Sample: 3, 3, 3, 3, 3, 7, 7, 7

$$X : \begin{pmatrix} 3 & 7 \\ \theta & 1-\theta \end{pmatrix}$$

(a) Method of moments: $\mu_1 = m_1 \iff 7 - 4\theta = 4.5 \iff 4\theta = 2.5 \iff \underline{\hat{\theta} = 0.625}$

$\mu_1 = E[X]$

m_1 – sample mean

$$\mu_1 = E[X] = 3\cdot\theta + 7(1-\theta) = 3\theta + 7 - 7\theta = 7 - 4\theta$$

$$m_1 = \frac{1}{8}(3+3+\dots+3+7+7+7) = \frac{36}{8} = 4.5$$

9.2. X - nr. of times a code is executed until it runs without errors

$X \sim Geom(p)$

Sample : 3, 7, 5, 3, 2

(a) Method of moments : $\mu_1 = m_1 \Leftrightarrow \frac{1}{p} = 4 \Leftrightarrow \hat{p} = \frac{1}{4} = \underline{\underline{0.25}}$

$\mu_1 = E[X]$

m_1 = sample mean

$X \sim Geom(p) \Rightarrow E[X] = \frac{1}{p}$

$m_1 = 4$